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Failure in distinguishing colored noise from chaos using the "noise titration" technique

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Identifying chaos in experimental data—noisy data—remains a challenging problem for which conclusive arguments are still very difficult to provide. In order to avoid problems usually encountered with techniques based on geometrical invariants (dimensions, Lyapunov exponent, etc.), Poon and Barahona introduced a numerical titration procedure which compares one-step-ahead predictions of linear and nonlinear models [Proc. Natl. Acad. Sci. U.S.A. **98**, 7107 (2001)]. We investigate the aformentioned technique in the context of colored noise or other types of nonchaotic behaviors. The main conclusion is that in several examples noise titration fails to distinguish such nonchaotic signals from low-dimensional deterministic chaos.

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I. INTRODUCTION

Time series from the real world very often exhibit a complex interplay between a deterministic and a stochastic components [1]. Special attention is thus required to distinguish any deterministic component within the signal. Many techniques have been proposed to detect chaos, but none is fully reliable. All such techniques rely on certain topological or information measures of attractors reconstructed from the data [2,3] and present some problems of specificity and reliability [4,5]. The largest Lyapunov exponent fails to distinguish chaos from noise [6].

A chaotic behavior is deterministic; that is, it can be described by differential, difference, or delay-differential equations. Determinism is the paradigm in which the future is determined by past and present events combined with a law of nature, as understood by Laplace [7]. Laplace's view fails for chaotic systems, for which it is no longer possible to predict the future for an infinite time. Due to this, chaos in experimental data cannot always be distinguished from stochasticity using statistical analysis [8] and identifying determinism remains a challenge. Glass clearly pointed out that prior to asserting that some dynamics is chaotic, there should be clear evidence of determinism [9].

By underlying determinism we mean low-dimensional determinism, since high-dimensional determinism cannot be distinguished from a stochastic process in general. Proving that "low-dimensional" chaotic dynamics underlies a short noisy time series is a most difficult problem to address. In this respect surrogate data analysis has been used often [10]. Unfortunately, such techniques only test whether the investigated time series can be distinguished from surrogate data, or not. This is therefore not a direct—and definite—answer to the original question of detecting determinism.

Assuming that determinism has been detected, usually it is desirable to go a step further and try to establish if the data were produced by a dynamical process which is bounded, sensitive to initial conditions, and recurrent. The first feature is certainly the one that involves less risk. Sensitivity to initial conditions is usually established computing the largest Lyapunov exponent, although such a computation is still a great challenge for short and noisy time series [1]. Finally, by definition, recurrence—which is related to the population of unstable periodic orbits around which chaotic behaviors are organized [11]—can only be tested for long time series. The relative organization of periodic orbits leads to the architecture of chaotic attractors [12]. Getting periodic orbits from short time series necessarily requires the estimation of a global model which can then be integrated over a long time (see [13–16] among others). One of the strongest pieces of evidence of determinism underlying a data set is a valid global model obtained from the data. When the measured time series is sufficiently long, topological analysis [17,18] is probably the most exacting validation procedure. When the time series is short, other procedures should be considered [19–21].

Techniques for detecting the presence of nonlinear determinism in experimental data have been discussed in [22,23]. A technique referred to as "titration of chaos" based on numerical titration of the data was proposed in [24]. More recently, the method has been called "noise titration" [25,26]. Unlike the surrogate data technique, noise titration has been far less investigated in the literature, despite its claims and promises. It is the objective of this Rapid Communication to provide conclusions of such an investigation and to report that noise titration has been found to incorrectly classify nondeterministic systems as chaotic and to be generally inadequate to distinguish between low-dimensional chaos and noise.

After a brief description of the noise titration technique (Sec. II), two examples where this technique fails are discussed in Sec. III before stating the main conclusions in Sec. IV.

II. TITRATION OF CHAOS

Poon and Barahona's method of chaos detection [24]—that claims to be able to robustly test for the presence of deterministic chaos in short, noisy time series—is based on two steps: a nonlinear detection method [27] and gradual addition of noise. The method is as follows. Linear and nonlinear models are obtained for the data under test. The method follows by verifying, within a certain confidence interval, which model class better describes the data. Afterward, white or linearly correlated noise is added to the data. The standard deviation of the added noise is gradually increased until the nonlinear model is not any better than the linear one. At this point it is said that the method failed to detect any nonlinearity. The standard deviation of the added noise, σ_e, at this point, divided by the standard deviation of the data, σ_y, multiplied by 100 is called the noise limit (NL) and was interpreted as a measure of chaos in the data [24]. The condition NL$>$0—in other words, the condition that a certain amount of noise must be added in order for nonlinearity not to be detected or, equivalently, nonlinearity *is* detected in the original data—is claimed to be sufficient to confirm the presence of chaos.

In order to better understand the "titration-of-chaos" method, some questions should be answered. First, what is the motivation for adding noise to the data? The name "titration" comes from the analogy with the chemical process where one measures the acid concentration in a solution by gradually adding an alkaline solution of known concentration until neutralization. As the volume of the base solution required to neutralize the acid is a measure of the acid's concentration, so the noise variance required to prevent nonlinearity detection would be a measure of the chaoticity of the data. However, if one is only interested in knowing whether or not there is chaos, the proposed condition amounts to NL$>$0. Technically speaking, the test for chaos does not need the addition of noise.

The concept of titration is nice because noise is always present in measured data and it becomes quite difficult to provide a definite "yes-or-no" answer. Such a feature leads to the distinction between noisy chaos and chaotic noise [1]. Another important question is whether or not NL$>$0 really implies chaos. In [24], only one kind of system is studied: deterministic nonlinear systems (chaotic or otherwise) with measurement (additive) noise. It is shown that, for the particular type of systems studied, the NL value is correlated with the largest Lyapunov exponent (λ_{max}), at least in the regions where λ_{max} is positive. Moreover, the conclusion that NL$>$0 implies chaos is drawn from the analysis of how the method behaves with these examples and nowhere a formal proof of the claim is given. Does this conclusion remain valid for other classes of systems? The next section will provide a negative answer for this question.

III. NONCHAOTIC CASES WITH NL$>$0

Consider the sine map $x_{n+1}=\mu \sin(x_n)$, where μ is a bifurcation parameter. This map is equivariant since applying $x_n \mapsto -x_n$ leads to $x_{n+1} \mapsto -x_{n+1}$. This means that solutions to the sine map present some properties induced by an order-2 symmetry. For instance, when $\mu \in [2;3.1]$, there are two coexisting symmetry-related solutions, mapped to each other by $x_n \mapsto -x_n$. When μ is increased, two simultaneous period-doubling cascades are observed as a route to chaos.

A deterministic nonchaotic randomly driven dynamics is obtained using

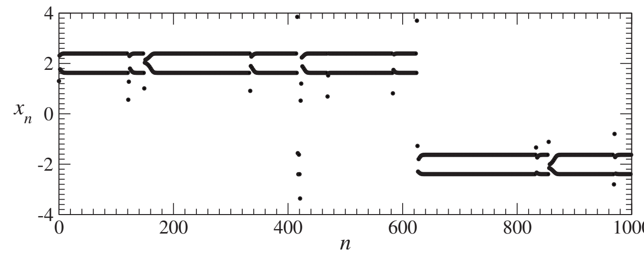

FIG. 1. Time series (1000 points) produced by map (1). Parameter values: μ=2.4, b=2, and q=0.01.

$$x_{n+1} = \mu \sin(x_n) + Y_n \eta_n, \quad (1)$$

with (μ=2.4) for which a period-2 limit cycle is produced and where Y_n is a random variable from a Bernoulli process and η_n is an independently and identically distributed (iid) random variable with a uniform distribution between $-b$ and b. The value of each Y_n is 1 with probability q and 0 with probability $1-q$. When q is small, stochastic perturbations are quite rare and the dynamics produced by (1) and the original sine map are quite similar.

One thousand iterations of map (1) were produced using q=0.01, μ=2.4, and b=2 (Fig. 1). The resulting behavior is roughly a period-2 limit cycle, randomly destabilized by the stochastic perturbations $Y_n\eta_n$, which are sometimes sufficient to send the trajectory to the other symmetric solution, as seen around iteration 620 in Fig. 1.

With q=0.01, only 11 perturbations occurred. After each perturbation, there is a short transient after which the trajectory settles onto one of the two coexisting period-2 limit cycles. The first-return map (Fig. 2) shows two truncated parabola with some points randomly distributed around. "Parabolic" shapes are visited during transient regimes. Obviously, this noisy periodic dynamics is not chaotic, but it *is* nonlinear. This is confirmed by the largest Lyapunov exponent, which is negative (λ_{max}=$-$0.65).

According to [24], white (or linearly correlated) noise of increasing variance σ_e^2 is added to the data until any potential nonlinearity goes undetected. This is determined by predict-

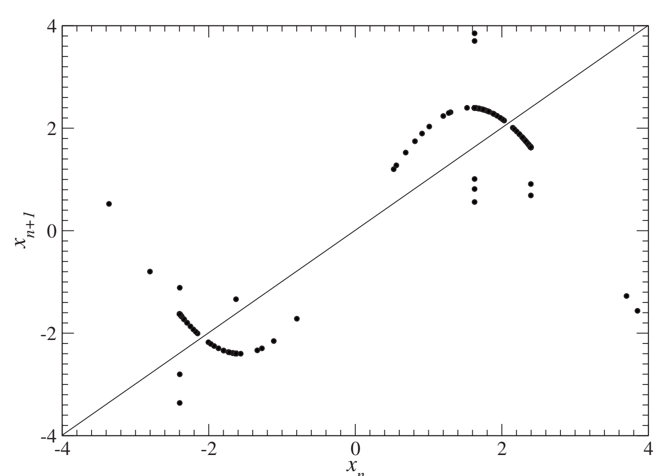

FIG. 2. First-return map computed from the trajectory solution to map (1).

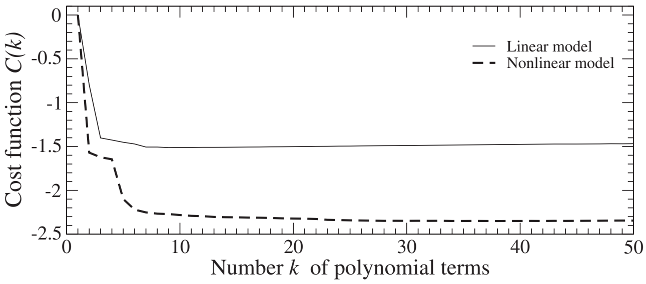

FIG. 3. Cost function $C(k)$ for linear and nonlinear models for no noise added ($\sigma_e=0$). Linear and nonlinear autoregressive polynomial models of increasing number of terms k were fitted to the data. The cost function is $C(k)=\log_e \epsilon(k)+\frac{k}{N}$, where k is the number of terms, $\epsilon(k)$ is the error, and N is the length of the time series [27]. The nonlinear models are better predictors than the linear counterparts regardless the value of k.

ing the noisy data with both linear and nonlinear models for increasing σ_e^2. Assuming nonlinearity in the data, the nonlinear models will outperform the linear ones up to a limiting value of σ_e which is used to obtain the NL beyond which there is no advantage in using nonlinear models for prediction. Based on this scheme, Poon and Barahona would have claimed that NL$>$0 indicates chaos and the value of NL gives an estimate of its relative intensity. If such an assumption was true, a NL$\approx$0 should be expected in the previous example, since the dynamics studied is not chaotic. However, NL=20% is obtained; that is, when noise with standard deviation $\sigma_e=20\sigma_y/100$ is added, nonlinear models perform similar to linear ones. This would have been incorrectly interpreted to indicate that the underlying dynamics is chaotic because the original data are better predicted by a nonlinear model than by a linear one for any reasonable model size (Fig. 3). This is therefore a first example where the noise titration technique fails by giving a false positive detection of chaos. Such a failure was not reported by Poon and Barahona because they always investigated purely deterministic dynamics additively contaminated by noise. In a recent paper different types of noise were investigated and pitfalls of the noise titration procedure were pointed out [28].

In order to produce a nonlinearly correlated noise, random noise is used to drive a nonlinear moving average filter as

$$x_{n+1} = a\nu_n + b\nu_{n-1}(1-\nu_n), \tag{2}$$

where ν_n is a uniform iid random variable with values between 0 and 1. This random signal is nonlinear colored noise. Its stochastic character is made evident by its first-return map (Fig. 4). No deterministic structure (like a parabolic shape or other) can be found in Fig. 4. Since the dynamics underlying the nonlinear moving average model (2) is obviously sensitive to initial conditions, the largest Lyapunov exponent is positive.

Applying noise titration to this stochastic solution to map (2) leads to a noise limit NL=35%. According to Poon and Barahona, this would indicate that these data correspond to chaos with an intensity greater than in the previous example. Again this would be a wrong conclusion. What is titrated is,

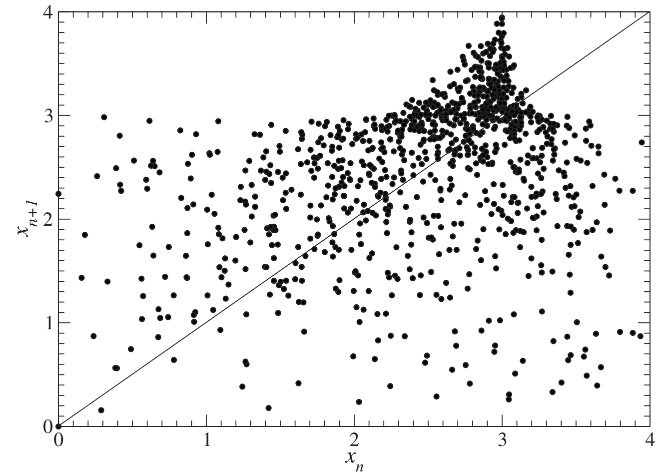

FIG. 4. First-return map computed from a trajectory produced by (2).

in fact, the action of the nonlinearity on the sensitivity to initial conditions.

Consequently, once again, this technique would erroneously conclude in favor of a chaotic deterministic behavior, although the underlying dynamics is clearly not deterministic. The reason is similar as for the previous case; that is, this nonlinear colored noise is predicted more accurately by nonlinear models than by linear ones.

Finally, we search for a purely chaotic dynamics for which the noise limit was also about 35%. In order to do so, the logistic map

$$x_{n+1} = \mu x_n(1-x_n) \tag{3}$$

was investigated with increasing values for parameter μ. It was finally found that for $\mu=3.62$, the noise limit was about 35%. For this μ value, the first-return map looks like a two-banded parabola (Fig. 5). This means that in a blind test, the noise titration does not differentiate between nonlinear colored noise (Fig. 4) and a purely chaotic behavior (Fig. 5).

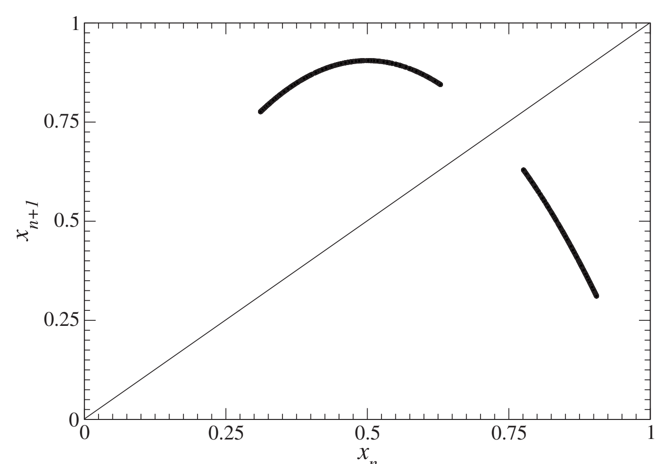

FIG. 5. First-return map computed from a trajectory produced by the logistic map (3). Parameter value: λ=3.62.

The previous examples show that noise titration fails when there is no underlying determinism by indicating chaos, which is an impossibility, since there is no determinism. This was never reported before because the original paper only investigated deterministic dynamics.

IV. CONCLUSION

Asserting the presence of chaos in experimental data is a rather difficult problem. In addressing this, low dimensionality is an important aspect, because it is always possible to assume that natural processes are deterministic by definition [26]. But such an assumption has an unavoidable metaphysical character. Indeed, offering a proof for determinism is one of the most difficult tasks one faces when experimental data are investigated. Global modeling is an exacting technique [29], although it is observable dependent [30]. As with other nonlinear detection schemes, when the objective is to decide whether a dynamics is chaotic or not, noise titration also fails under certain circumstances. In other words, whereas the condition $\mathrm{NL}>0$ is helpful to detect nonlinearity as proposed in [27], it is *not* a sufficient condition for chaos, as suggested in [24]. Indeed, noise titration tells us more about the nonlinear character of dynamical processes than about "chaos intensity" as shown by the examples discussed in this Rapid Communication.

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